Numerical Analysis Preliminary Exam

August 15, 2011

Solutions

1. **Quadrature**

The Chebyshev polynomials of the second kind are defined as

$$U_n(x) = \frac{1}{n+1} T'_{n+1}(x); \quad n \geq 0;$$

where $T_{n+1}(x)$ is the Chebyshev polynomial of the first kind.

(a) Using the form $T_n(x) = \cos(n\theta); \quad x = \cos(\theta); \quad x \in [-1; 1];$ derive a similar expression for $U_n(x)$.

(b) Show that the Chebyshev polynomials of the second kind satisfy the recursion

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1} \end{aligned}$$

(c) Show that the Chebyshev polynomials of the second kind are orthogonal with respect to the inner product

$$< f; g >= \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}dx.$$

(d) Derive the 3 point Gauss Quadrature rule for the integral

$$I_3(f) = \sum_{j=1}^{3} w_j f(x_j) = \int_{-1}^{1} f(x)\sqrt{1-x^2}dx + E_3(f);$$

Solution:

(a) Using the expression $T_n(x) = \cos(n\theta);\quad x = \cos(\theta);\quad x \in [-1; 1];$ we have

$$T'_{n+1}(x) = -(n+1)\sin((n+1)\theta)\frac{d\theta}{dx} = (n+1)\frac{\sin((n+1)\theta)}{\sin(\theta)};$$

which yields

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)};\quad x = \cos(\theta);\quad x = [-1; 1].$$

(b) The first two terms are found by definition. The recursion is established by noting

$$\begin{aligned} U_{n+1}(x) + U_{n-1}(x) &= \frac{\sin((n+2)\theta)}{\sin(\theta)} + \frac{\sin((n)\theta)}{\sin(\theta)} \\ &= \frac{\sin(((n+1)+1)\theta)}{\sin(\theta)} + \frac{\sin(((n+1)-1)\theta)}{\sin(\theta)} \\ &= 2\cos(\theta)\frac{\sin((n+1)\theta)}{\sin(\theta)} = 2xU_n(x). \end{aligned}$$

Substituting $x = \cos(\theta)$ yields the result.

(c) Unsing the trig substitution, $x = \cos(\theta)$, we have

$$\begin{aligned} < U_n; U_m > &= \int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx \\ &= \int_0^{\pi} \frac{\sin((n+1)\theta)}{\sin(\theta)}\frac{\sin((m+1)\theta)}{\sin(\theta)}\sin^2(\theta)d\theta \\ &= \int_0^{\pi} \sin((n+1)\theta)\sin((m+1)\theta)d\theta \\ &= 0 \quad \text{for } n \neq m. \end{aligned}$$

(d) The quadrature points are the roots of $U_3(x) = 0$, which can be found either using the recursion to derive $U_3(x) = 8x^3 - 4x$ or by setting $\sin(4\theta) = 0$ which yields $\theta_j = k\pi/4$ for $j = 1; 2; 3$. The result is

$$x_1 = -1/\sqrt{2};\ x_2 = 0;\ x_3 = 1/\sqrt{2}$$

The weights can be found by appealing to symmetry to imply $w_1 = w_3$. We also have

$$w_1 + w_2 + w_3 = \int_{-1}^{1}\sqrt{1-x^2}dx = \int_0^{\pi}\sin^2(\theta)d\theta = \pi/2$$

and

$$
\begin{aligned}
w_1 x_1^2 + w_3 x_3^2 &= \frac{w_1}{2} + \frac{w_3}{2} = w_1 = \int_{-1}^{1} x^2 \sqrt{1 - x^2}\, dx = \int_0^{\pi} \sin^2(\theta) \cos^2(\theta)\, d\theta \\
&= \int_0^{\pi} \frac{1 - \cos^2(2\theta)}{4} d\theta = \int_0^{\pi} \frac{1 + \cos(4\theta)}{8} d\theta = \pi/8.
\end{aligned}
$$

2. Linear Algebra

(a) Descibe the singular value decomposition (SVD) of the m n matrix A. Include an explanation of the rank of A and how the SVD relates to the four fundamental subspaces

$R(A)$	Range of A	$R(A\ )$	Range of A
$N(A)$	Nullspace of A	$N(A\ )$	Nullspace of A

(b) Perform the SVD on the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) Compute the pseudo-inverse of A (the Moore-Penrose pseudo-inverse) Leave in factored form.

(d) Find the minimal-length least-stdrr(factored)]TJ -174.385 -14.44621(u065eetored)]TJ -15 -14.4462oi51

Solution:

(a) Any m n matrix can be decomposed as

$$A = U \quad V \; ;$$

where U is an m m unitary matrix, V is an n n unitary matrix and is an m n diagonal matrix containing the singular values of A. Denote them as

$$_1 \qquad _2 \quad ::: \; _n \qquad 0$$

Assume that U and V have been constructed so the singular values are ordered as above. Let r be the smallest index for which $_r > 0$. Then r is the rank of A.

The columns of V are the right singular vectors of A and the columns of U are the left singular vectors of A. The columns of U corresponding to nonzero singular values span $R(A)$ while the columns of U corresponding to zero singular values span $N(A\;)$. The columns of V corresponding to nonzero singular values span the $R(A\;)$ while the columns of V that correspond to zero singular values span $N(A)$. That is

$$\begin{aligned} R(A) &= \text{span} f\underline{u}_1; \underline{u}_2; :::; \underline{u}_r g \\ N(A\;) &= \text{span} f\underline{u}_r; \underline{u}_{r+1}; :::; \underline{u}_m g \\ R(A\;) &= \text{span} f\underline{v}_1; \underline{v} \end{aligned}$$

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This yields

$$\underline{u}_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} \qquad \underline{u}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

The third left singular vector is the null space of A , which yields

$$\underline{u}_3 = \begin{pmatrix} 1/\sqrt{18}\,p \end{pmatrix}$$

3. **Eigenvalues** Define the $k \times k$ tridiagonal matrix

$$T_k = \begin{bmatrix} a_1 & b_2 & & & \\ c_2 & a_2 & b_3 & & \\ & c_3 & a_3 & \ddots & \\ & & \ddots & \ddots & b_k \\ & & & c_k & a_k \end{bmatrix}.$$

The characteristic polynomial of T_k is given by $p_k(\lambda) = det(\lambda I - T_k)$.

(a) Define $p_k(\lambda)$ in terms of $p_{k-1}(\lambda)$ and $p_{k-2}(\lambda)$.

(b) Show that if $c_j b_j > 0$ for $j = 2, \ldots, k$, then $p_k(\lambda) = 0$ has only real roots. (Hint: find a real similarity transformation that symmetrizes T_k.)

(c) Assume $c_j b_j > 0$ for $j = 2, \ldots, k$ and assume that the roots of $p_{k-2}(\lambda)$ separate the roots of $p_{k-1}(\lambda)$, that is, between each adjacent pair of roots of $p_{k-1}(\lambda)$, there is a root of $p_{k-2}(\lambda)$. Prove that the roots of $p_{k-1}(\lambda)$ separate the roots of $p_k(\lambda)$. (Hint: draw a picture and use the recursion.)

Solution:

(a) Expanding the last column of $\det(\lambda I - T_k)$ yields

$$p_k(\lambda) = (\lambda - a_k)p_{k-1}(\lambda) - b_k c_k p_{k-2}(\lambda).$$

(b) Let $r_i = \sqrt{b_j/c_j}$, $d_1 = 1.0$ and $d_j = r_j d_{j-1}$ for $j > 1$. Define the matrix $D_k = diag(d_1, \cdots, d_j, \cdots, d_k)$. This yields

$$D_k T_k D_k^{-1} = \begin{bmatrix} a_1 & \sqrt{b_2c_2} & & & \\ \sqrt{b_2c_2} & a_2 & \sqrt{b_3c_3} & & \\ & \sqrt{b_3c_3} & a_3 & \ddots & \\ & & \ddots & \ddots & \sqrt{b_kc_k} \\ & & & \sqrt{b_kc_k} & a_k \end{bmatrix}.$$

(c) Denote the roots of $p_\ell(\lambda)$ by $\lambda_1^\ell < \lambda_2^\ell, \ldots, < \lambda_\ell^\ell$. Using the recursion derived above we see that

$$\begin{aligned} p_k(\lambda_j^{k-1}) &= -b_kc_kp_{k-2}(\lambda_j^{k-1}) \\ p_k(\lambda_{j+1}^{k-1}) &= -b_kc_kp_{k-2}(\lambda_{j+1}^{k-1}) \end{aligned}$$

Since the roots of $p_{k-2}(\lambda)$ separate the roots of $p_{k-1}(\lambda)$, we have $p_{k-2}(\lambda_j^{k-1})p_{k-2}(\lambda_{j+1}^{k-1}) < 0$ and conclude that $p_k(\lambda)$ must have at least one root between each root of $p_{k-1}(\lambda)$.

Since each $p_\ell(\lambda)$ is monic and λ_{k-1}^{k-1} is greater than all the roots of $p_{k-2}(\lambda)$, then we may conclude that $p_{k-2}(\lambda_{k-1}^{k-1}) > 0$. Consider the equation

$$p_k(\lambda_{k-1}^{k-1}) = -b_kc_kp_{k-2}(\lambda_{k-1}^{k-1}) < 0.$$

Since $p_k(\lambda)$ is also monic, this implies that $p_k(\lambda)$ has a root greater than λ_{k-1}^{k-1}. A similar argument shows the $p_k(\lambda)$ has a root less than λ_1^{k-1}.

Thus, the roots of $p_{k-1}(\lambda)$ separate the roots of $p_k(\lambda)$.

Solution:

(a) Applying Newton's method to $f(x) = x^2 - c$ gives
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

5. ODE

The Forward Euler (FE) method for solving

$$y'(t) = f(t, y(t)),\ \ y(t_0) = y_0 \tag{5.1}$$

uses for each step the first two terms of its Taylor expansion, i.e.

$$y(t+h) = y(t) + h f(t, y(t)). \tag{5.2}$$

The *Taylor Series Method* generalizes (5.2) to include further terms in the expansion

$$y(t+h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \ldots + c_n h^h \ \ (+O(h^{n+1})). \tag{5.3}$$

The main interest in the Taylor series method arises when one wants extremely high orders of accuracy (typically in the range of 10-40). There are three main ways to determine (in each step) the constants $c_0,\ c_1,\ c_2, \ldots$ Many numerical text books consider only the first procedure listed below (and then dismiss the Taylor approach as generally impractical, since the number of terms more than doubles by each iteration):

Procedure 1: Differentiate (5.1) repeatedly to obtain

$$\begin{aligned} y' &= f \\ y'' &= f\tfrac{\partial f}{\partial y} + \tfrac{\partial f}{\partial t} \\ y''' &= f^2 \tfrac{\partial^2 f}{\partial y^2} + f\left\{\left(\tfrac{\partial f}{\partial y}\right)^2 + 2\tfrac{\partial^2 f}{\partial t \partial y}\right\} + \left\{\tfrac{\partial^2 f}{\partial t^2} + \tfrac{\partial f}{\partial t}\tfrac{\partial f}{\partial y}\right\} \\ &\ldots. \end{aligned} \tag{5.4}$$

and then use $c_k = y^{(k)}(t)/k!$

Consider next the special case of (5.1) $y' = t^2 + y^2$. Find the first three coefficients $c_0,\ c_1,\ c_2$, starting from a general point t by means of the approaches suggested in parts (a) - (c) below. (Needless to say, you should get the same answer in all three cases)

(a) Use *Procedure 1*, as described above.

(b) Use *Procedure 2*: Note that (5.1) implies

$$\frac{d\,y(t+h)}{dh} = f(t+h, y(t+h)). \tag{5.5}$$

Substitute some leading part of (5.3) into (5.5) and equate coefficients.

(c) Use *Procedure 3*: Note that the first term of (5.3) is known. After that, each time a truncated version of (5.3) is substituted into the right hand side (RHS) of (5.5) and integrated, one gains

Solution:

(a) Immediate use of $y' = f,\ y'' = f\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}$ gives

$$y(t+h) = y(t) + h(t^2 + y(t)^2) + \tfrac{1}{2}h^2((t^2 + y(t)^2)2y(t) + 2t) = y(t) + h(t^2 + y(t)^2) + h^2((t^2 + y(t)^2)y(t) + t).$$

(b) Substituting the expression $y(t+h) = c_0 + c_1 h + c_2 h^2 + \ldots$ into $\dfrac{dy(t+h)}{dh} = f(t+h, y(t+h))$ gives

$$c_1 + 2hc_2 + \ldots = (t+h)^2 + (y(t) + c_1 h \neq_1 (h\chi +$$

6. PDE

The standard second order finite difference approximation to the ODE $u''(x) = f(x)$ can schematically be written as

$$[1 - 2 \;\; 1]\, u/h^2 = [1]\, f + O(h^2) \tag{6.1}$$

(a) Verify that the approximation

$$[1 - 2 \;\; 1]\, u/h^2 = [1 \;\; 10 \;\; 1]\, f/12 + O(h^4) \tag{6.2}$$

indeed is fourth order accurate.

The 2-D counterparts to (6.1) and (6.2) for approximating the Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ are

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \frac{u}{h^2} = [1]\, f + O(h^2) \tag{6.3}$$

and

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \frac{u}{6h^2} = \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} \frac{f}{12} + O(h^4), \tag{6.4}$$

respectively.

(b) Sketch the structure and give the entries of the linear system that is obtained when we use (6.4) to solve a Poisson equation with Dirichlet boundary conditions on the square domain $[0, 1]$ % $[0, 1]$.

(c) In the case when $f(x, y)$ h 0 (i.e. solving Laplace's equation), we would expect from (6.3) and (6.4) that

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \frac{u}{h^2} = O(h^2) \tag{6.5}$$

and

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \frac{u}{6h^2} = O(h^4). \tag{6.6}$$

This is correct for (6.5) but (remarkably), the accuracy of (6.6) now jumps to $O(h^6)$. Without working through the details, outline an approach for verifying this increased order of accuracy.

Solution:

(a) Taylor expansion around x gives
$[1 - 2 \ \ 1]\, u/h^2 - [1 \ \ 10 \ \ 1]\, f/12 \ = \ \{u(x-h) - 2u(x) + u(x+h)\}/h^2 - \{f(x-h) + 10 f(x) + f(x+h)\}/12 =$
$\{u''(x) + \tfrac{1}{12}h^2 u^{(4)}(x) + O(h^4)\} \ - \{f(x) + \tfrac{1}{12}h^2 f''(x) + O(h^4)\}.$
With $u'' = f,$ it also holds that $u^{(4)} = f''$. Therefore, the expression above reduces to $O(h^4)$.

(b) See next page.

(c) Similar to part a, immediate Taylor expansion would give

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \frac{u}{6h^2} = A + B h^2 + C h^4 + D h^6 + \ldots$$

where each of the expressions $A, B, C, D, \ldots$ would be partial derivative operators, applied to u at the origin. For the stated result to hold, it would be required that

$$A = \left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right) u$$

and that the operators for B and C both can be factored so that a factor $\left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right)$ emerges. This would ensure they evaluate to zero whenever u satisfies $\tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2} = 0.$

If one really works this out, it will transpire that:

$$A = \left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right) u \ \ ,$$

$$B = -\left(\tfrac{1}{12}\tfrac{\partial^4}{\partial x^4} + \tfrac{1}{6}\tfrac{\partial^4}{\partial x^2 \partial y^2} + \tfrac{1}{12}\tfrac{\partial^4}{\partial y^4}\right) u = -\tfrac{1}{12} \qquad\qquad ,$$

(b)

Let u_{ij} denote the i,j entry of u, i.e. $u(x_i, y_j)$,
f_{ij} denote $f(x_i, y_j)\,6h^2/12$,
b_{ij} denote $b(x_i, y_j)$,
then the relations take the form

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\end{array}\right]
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With everything in the RHS known, all entries of u can be found by solving the symmetric, positive definite linear system above.