
Formulate Newton's method for solving the nonlinear $\underline{2} \times 2$ system of equations

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned}$$

In the same style as how one proves quadratic convergence in the scalar case for $f(x) = 0$, show quadratic convergence (assuming sufficient smoothness of f, g , root being simple, etc.) in the $\underline{2} \times 2$ case. Assuming the root $x = \alpha, y = \beta$ to be of multiplicity one, define $e_n = x_n - \alpha, v_n = y_n - \beta$, and show that both e_{n+1} and v_{n+1} are of size $O(e_n^2, v_n^2)$.

We first recall the proof for the scalar case $f(x) = 0$

(1) Consider quadrature

$$(0.1) \quad I_{quad} = \sum_{i=0}^n \alpha_i f(x_i), \quad x_i \in [-1, 1]$$

for the integral

$$I = \int_{-1}^1 f(x) w(x) dx,$$

where w is a positive weight in $(-1, 1)$. Let

$$\Omega_{n+1}(x) = \prod_{i=1}^n (x - x_i)$$

denote the polynomial of degree $n+1$ associated with the (distinct) quadrature nodes x_0, x_1, \dots, x_n .

$$I - I_{quad} = \int_{-1}^1 \Omega_{n+1}(x) p(x) w(x) dx$$

is zero if and only if the quadrature is exact for any polynomial of degree less or equal to n .

Proof:

(1) If the quadrature formula (0.1) is exact for all polynomials of degree less or equal to n , then

$$\int_{-1}^1 \Omega_{n+1}(x) p(x) w(x) dx = \sum_{i=0}^n \alpha_i \Omega_{n+1}(x_i) p(x_i) w(x_i) = 0$$

Let us consider polynomial $f(x)$ of degree less or equal to n . We can write

$$f(x) = \Omega_{n+1}(x) \pi_{m-1}(x) + q_n(x),$$

$$I = \int_{-1}^1 f(x) w(x) dx$$

$$= \int_{-1}^1 q_n(x) w(x) dx$$

and, by the direct evaluation,

$$\begin{aligned} I &= \sum_{i=0}^n \alpha_i f(x_i) \\ &= \sum_{i=0}^n \alpha_i \Omega_{n+1}(x_i) \pi_{m-1}(x_i) + \sum_{i=0}^n \alpha_i q_n(x_i) \\ &= \sum_{i=0}^n \alpha_i q_n(x_i). \end{aligned}$$

We can then conclude by observing that the quadrature weights α_i can always be chosen to satisfy

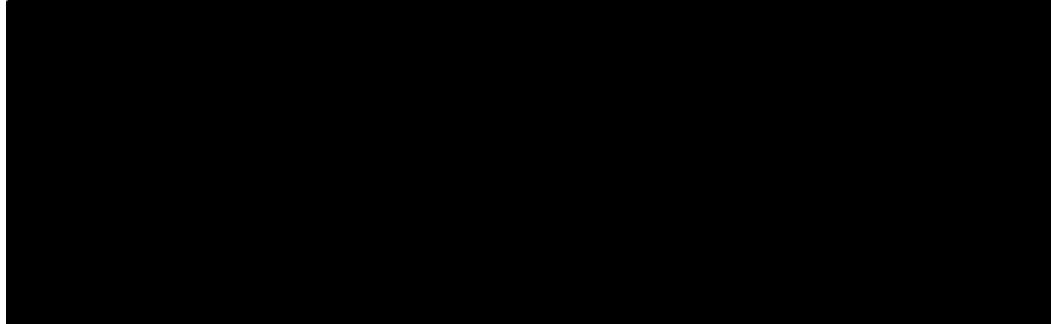
$$I = I_{quad}$$

for an arbitrary polynomial of degree less or equal to n .

Assuming that p_n , $n = 0, 1, 2, \dots$ form a set of orthogonal polynomials of degrees n over some interval $[a, b]$ with weight function $w(x) > 0$, show that they obey a three-term recursion relation of the form

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x), \quad n = 1, 2, 3, \dots$$

Let $A \in \mathbb{C}^{n \times n}$ be a symmetric complex valued matrix, $A = A^t$. It is possible to



Proof:



$$Au = \mu u,$$

we also have

$$A\bar{u} = \mu\bar{u}$$

as well as

$$A\bar{A}u = \mu A\bar{u} = \mu^2 u.$$

$$\bar{A}A\bar{u} = \mu\bar{A}u = \mu^2 \bar{u}.$$

are the singular vectors and thus they are orthonormal. We recognize that u and \bar{u} a



Consider the 4th order Adams-Bashforth scheme (AB4) for solving the ODE $y' = f(x, y)$:

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}).$$

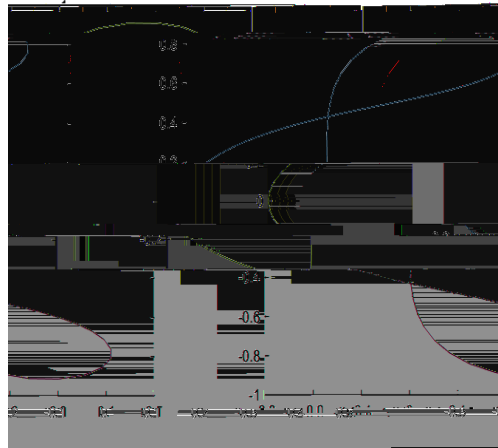
a. Apply the *root condition* to this scheme. Explain the outcome of the test, and explain what information this provides regarding the scheme.

b. The Matlab code

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r = exp(complex(0, linspace(0, 2*pi)));  
i = 24*(r.^4 - r.^3) ./ (55*r.^3 - 59*r.^2 + 37*r - 9);  
plot(i);
```

generates the figure shown to the right.

- i. Derive the relation used in the code.
- ii. Explain (no need to do the algebra) how you



ii. The generated curve marks all possible ω -values for when a root r is on the periphery of the unit circle. If ω crosses a curve segment, a root moves between the inside and the outside of the unit circle. The stability

